2/4/26

Quiz 4 (10 minutes)

§2.2 starts talking about more specific (and important) linear transformations. We started discussing one of them. Finish that discussion, then give an overview of the rest.

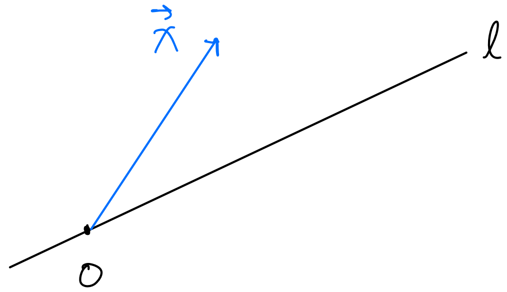
Orthogonal Projections in the Plane $\mathbb{R}^2$

We 1st gave a geometric description, then started looking at the algebra.

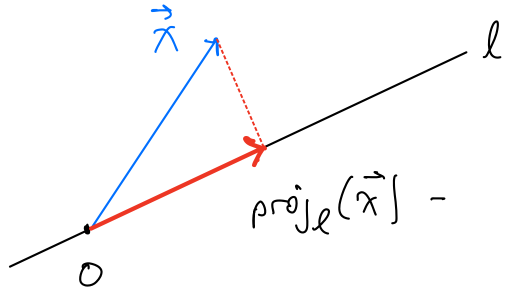
1st summarize what we already did:

Let ℓ be a line in the plane $\mathbb{R}^2$ thru the origin. (We'll see $\mathbb{R}^2$ isn't necessary.)

Let $\vec{x}$ be any vector in $\mathbb{R}^2$



We defined the orthogonal projection, $\text{proj}_\ell(\vec{x})$, of $\vec{x}$ to ℓ:

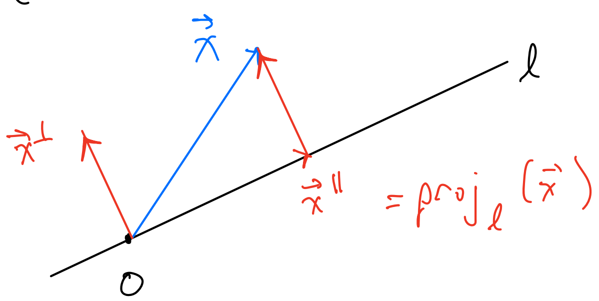


To find $\text{proj}_\ell(\vec{x})$ we addressed a related problem:

Problem: How can we write $\vec{x}$ as a sum of vectors

$$\boxed{\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}} \qquad (*)$$

where $\vec{x}^{\parallel}$ is parallel to ℓ (specifically it lies on ℓ) and $\vec{x}^{\perp}$ is perpendicular to ℓ?



Step 1 Choose any point W on ℓ. Let $\vec{w}$ be the vector defined by W. Our desired vector $\vec{x}^{\parallel}$ is some scalar multiple of $\vec{w}$, say $\vec{x}^{\parallel} = k\vec{w}$. What's k?

Step 2 We computed last time:

$$k = \frac{\vec{x}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}$$

Putting it together:

Step 3 $$proj_\ell(\vec{x}) = \vec{x}^{\parallel} = k\vec{w} = \left(\frac{\vec{x}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w} = \frac{(\vec{x}\cdot\vec{w})}{\|\vec{w}\|^2}\vec{w}$$

Now, our $\vec{w}$ was <u>any</u> vector lying on ℓ. Suppose we picked $\vec{w}$ to have length 1, ie $\vec{w}$ is a unit vector. Call it $\vec{u}$ now, for emphasis.

Then $$proj_\ell(\vec{x}) = \vec{x}^{\parallel} = \left(\frac{\vec{x}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u}.$$

But $\vec{u}\cdot\vec{u} = \|\vec{u}\|^2 = 1$ so

$$\boxed{proj_\ell(\vec{x}) = (\vec{x}\cdot\vec{u})\,\vec{u}}$$

In fact we already know what $\vec{u}$ is, in terms of $\vec{w}$:

<u>lemma</u> if $\vec{w}$ is any vector in $\mathbb{R}^n$ then $\dfrac{\vec{w}}{\|\vec{w}\|}$ is a unit vector

<u>proof</u>

For convenience write $c = \|\vec{w}\|$. Say the coordinates of $\vec{w}$ are $w_1, \dots, w_n$, so the coords of $\dfrac{\vec{w}}{\|\vec{w}\|}$ are $\dfrac{w_1}{c}, \dots, \dfrac{w_n}{c}$.

Then
$$\left\|\frac{\vec{w}}{c}\right\|^2 = \left(\frac{w_1}{c}\right)^2 + \dots + \left(\frac{w_n}{c}\right)^2$$
$$= \frac{1}{c^2}\left(w_1^2 + \dots + w_n^2\right)$$
$$= \frac{1}{c^2}\|\vec{w}\|^2$$

So $\left\|\dfrac{\vec{w}}{c}\right\| = \dfrac{1}{c}\|w\| = 1$. //

The book uses this in the solution of Example 3.

Back to $\mathbb{R}^2$:

Conclusion:

$$\boxed{\mathrm{proj}_\ell(\vec{x}) = (\vec{x}\cdot\vec{u})\,\vec{u} \quad \text{where} \quad u = \frac{1}{\|\vec{w}\|}\vec{w}}$$

this should be memorized

<u>Important observation</u>

We know $(\vec{x}\cdot\vec{u}) = (\|\vec{x}\|)(\|\vec{u}\|)(\cos\theta)$

$= \|\vec{x}\|\cos\theta$ where θ is the angle between $\vec{x}$ and $\vec{u}$.

None of this depends on being in $\mathbb{R}^2$. So if ℓ is a line in $\mathbb{R}^n$ and $\vec{x} \in \mathbb{R}^n$, $\mathrm{proj}_\ell(\vec{x})$ makes sense and is given by the same formula.

We'll see $\mathbb{R}^3$ in a sec.

why is proj_ℓ a linear transformation? Now for convenience assume $\mathbb{R}^2$.

say $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. Note ℓ is not lost: it consists of all

← unit vector

the scalar multiples of $\vec{u}$.

then

$$\mathrm{proj}_\ell(\vec{x}) = (\vec{x}\cdot\vec{u})\,\vec{u} = (x_1u_1 + x_2u_2)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= \begin{bmatrix} u_1^2 x_1 + u_1u_2x_2 \\ u_1u_2x_1 + u_2^2x_2 \end{bmatrix}$$

$$= \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (\star)$$

So $\mathrm{proj}_\ell(\vec{x})$ is defined by mult. by this matrix! Note Σ diag $= 1$ since $\vec{u}$ is a unit vector.

This is summarized in Def 2.2.1.